Area inequalities for stable marginally trapped surfaces

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Abstract We discuss a family of inequalities involving the area, angular momentum and charges of stably outermost marginally trapped surfaces in generic non-vacuum dynamical spacetimes, with non-negative cosmological constant and matter sources satisfying the dominant energy condition. These inequalities provide lower bounds for the area of spatial sections of dynamical trapping horizons, namely hypersurfaces offering quasi-local models of black hole horizons. In particular, these inequalities represent particular examples of the extension to a Lorentzian setting of tools employed in the discussion of minimal surfaces in Riemannian contexts.

1 Introduction

The Lorentzian nature of spacetime geometry, with its inherent notion of null cone, controls the rich features of light bending in general relativity. This includes in particular the possibility of causal disconnection between spacetime regions, as well as the convergence behavior of (trapped) light rays. Both aspects, related by the notion of (weak) cosmic censorship [44], lay at the basis of the concept of black hole in general relativity. In spite of the complexity of the generic situation, it is remarkable that stationary and vacuum black hole spacetimes are completely characterized by a few parameters with physical interpretation, namely the mass M (or, alternatively, the area A of the horizon), the angular momentum J and certain charges Q_i. These parameters fulfill a class of geometric inequalities that bound the mass by below. When using the horizon area A instead of the mass, they present the general form

$$(A/(4\pi))^2 \geq (2J)^2 + (\sum_i Q_i^2)^2 \ , \tag{1}$$

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and provide a family of inequalities written completely in terms of the quasi-local geometry of the black hole horizon. As a second remarkable point, these quasi-local geometric inequalities do extend to fully generic dynamical and non-vacuum situations, providing general lower bounds for the black hole horizon area.

In the stationary axisymmetric case with matter surrounding the horizon, these quasi-local inequalities have been proved to hold for the Killing horizon in [6, 40, 41, 5]. Regarding dynamical situations, and after the study in [25, 26, 27, 21, 22, 24, 23] of the global vacuum axisymmetric inequalities involving M, the quasi-local vacuum case has been studied in [28, 1, 33, 31] using axisymmetric initial data. Finally, in [43, 30, 34, 35] a purely spacetime (Lorentzian) perspective has been adopted, permitting to identify and refine the key assumptions, this leading to the extension of the inequalities to generic dynamical scenarios with matter. The study of these inequalities in higher dimensions has been started in [42]. A general review on geometric inequalities in axially symmetric black holes is presented in [29] .

Here we discuss these quasi-local inequalities, placing the emphasis on the involved Lorentzian aspects, namely the notion of stability of marginally outer trapped surfaces. The latter provide a Lorentzian analogue to the notion of stable minimal surfaces in Riemannian geometry. New results are presented regarding the incorporation in the inequalities of Yang-Mills charges and a geometric charge for certain divergence-free vectors on closed surfaces. We also comment on the interpretation of the integral characterization of the stability condition as an *energy flux* inequality.

2 Geometric and physical elements

Let (M, g_{ab}) be a 4-dimensional Lorentzian manifold satisfying Einstein equations

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \ , \tag{2}$$

where $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$ is the Einstein tensor associated with the Levi-Civita connection ∇_a, Λ is the cosmological constant and T_{ab} the stress-energy tensor. Unless otherwise stated, in the following the stress-energy tensor is assumed to satisfy a dominant energy condition (namely, given a future-directed causal vector v^a, then $-T^a{}_b v^b$ is a future-oriented causal vector) and the cosmological constant to be non-negative $\Lambda \geq 0$. We use throughout Penrose's abstract index notation (e.g. [51]).

2.1 Geometry of 2-surfaces

Let us consider a closed orientable 2-surface $\mathscr{S}$ embedded in (M, g_{ab}) (in the following, we shall assume that surfaces $\mathscr{S}$ are closed and orientable, unless otherwise stated). Let us denote the induced metric on $\mathscr{S}$ as q_{ab}, with Levi-Civita connection D_a, Ricci scalar ${}^2R$ and volume element ϵ_{ab} (we will denote by dS the

area measure on $\mathscr{S}$). Decomposing the tangent plane T_pM at each point $p \in \mathscr{S}$ as $T_pM = T_p\mathscr{S} \oplus T_p^{\perp}\mathscr{S}$, let us consider null vectors ℓ^a and k^a spanning the normal plane $T_p^{\perp}\mathscr{S}$ and normalized as $\ell^a k_a = -1$. This leaves a (boost) rescaling freedom $\ell'^a = f\ell^a$, $k'^a = f^{-1}k^a$. We can write

$$q_{ab} = g_{ab} + k_a\ell_b + \ell_a k_b \ . \tag{3}$$

Regarding the extrinsic curvature elements that we need in our analysis, let us consider the deformation tensors $\Theta^{(\ell)}_{ab}$ and $\Theta^{(k)}_{ab}$ along ℓ^a and k^a, respectively

$$\Theta^{(\ell)}_{ab} \equiv q^c{}_a q^d{}_b \nabla_c \ell_d \ \ , \ \ \Theta^{(k)}_{ab} \equiv q^c{}_a q^d{}_b \nabla_c k_d \ . \tag{4}$$

They determine the second fundamental form $\mathscr{K}^c_{ab}$ of $(\mathscr{S}, q_{ab})$ into (M, g_{ab}), namely $\mathscr{K}^c_{ab} \equiv q^d{}_a q^e{}_b \nabla_d q^c{}_e = k^c\Theta^{(\ell)}_{ab} + \ell^c\Theta^{(k)}_{ab}$ (cf. Senovilla's contribution in this volume). In particular, the expansion $\theta^{(\ell)}$ and the shear $\sigma^{(\ell)}_{ab}$ associated with the null normal ℓ^a, are given respectively by the trace and traceless parts of $\Theta^{(\ell)}_{ab}$

$$\theta^{(\ell)} \equiv q^{ab}\Theta^{(\ell)}_{ab} = q^{ab}\nabla_a\ell_b \ \ , \ \ \sigma^{(\ell)}_{ab} \equiv \Theta^{(\ell)}_{ab} - \frac{1}{2}\theta^{(\ell)} q_{ab} \ . \tag{5}$$

In addition, we consider the normal fundamental form $\Omega^{(\ell)}_a$

$$\Omega^{(\ell)}_a = -k^c q^d{}_a \nabla_d \ell_c \ , \tag{6}$$

that provides a connection on the normal bundle $T^*_{\perp}\mathscr{S}$. More specifically, considering a form $v_a \in T^*_{\perp}\mathscr{S}$, expressed as $v_a = \alpha\ell_a + \beta\ell_b$, we can write $q^c{}_a\nabla_c v_b = \Theta^{(v)}_{ab} + D^{\perp}_a v_b$, where $\Theta^{(v)}_{ab} = q^c{}_a q^d{}_b \nabla_c v_d = \alpha\Theta^{(\ell)}_{ab} + \beta\Theta^{(k)}_{ab}$ and

$$D^{\perp}_a v_b = D^{\perp}_a(\alpha\ell_a + \beta\ell_b) = (D_a\alpha + \Omega^{(\ell)}_a\alpha)\ell_b + (D_a\beta + \Omega^{(\ell)}_a\beta)k_b \ . \tag{7}$$

Transformation rules under a null normal rescaling $\ell'^a = f\ell^a$, $k'^a = f^{-1}k^a$ are

$$\theta^{(\ell')} = f\theta^{(\ell)} \ \ , \ \ \sigma^{(\ell')}_{ab} = f\sigma^{(\ell)}_{ab} \ \ , \ \ \Omega^{(\ell')}_a = \Omega^{(\ell)}_a + D_a(\ln f). \tag{8}$$

2.1.1 Axisymmetry

The introduction of a canonical angular momentum J on $\mathscr{S}$ requires imposing axisymmetry. In this context, we require the geometry of $\mathscr{S}$ to be axisymmetric with axial Killing vector η^a on $\mathscr{S}$. More specifically we require

$$\mathscr{L}_\eta q_{ab} = 0 \ \ , \ \ \mathscr{L}_\eta \Omega^{(\ell)}_a = 0 \ \ , \ \ \mathscr{L}_\eta \ell^a = \mathscr{L}_\eta k^a = 0 \ , \tag{9}$$

where η^a has closed integral curves, vanishes exactly at two points on $\mathscr{S}$ and is normalized so that its integral curves have an affine length of 2π. We adopt a tetrad

$(\xi^a, \eta^a, \ell^a, k^a)$ on $\mathscr{S}$, where the unit vector ξ^a tangent to $\mathscr{S}$ satisfies $\xi^a\eta_a = \xi^a\ell_a = \xi^a k_a = 0$, $\xi^a\xi_a = 1$. We can then write $q_{ab} = \frac{1}{\eta}\eta_a\eta_b + \xi_a\xi_b$, with $\eta = \eta^a\eta_a$, and

$$\Omega^{(\ell)}_a = \Omega^{(\eta)}_a + \Omega^{(\xi)}_a \quad , \quad \Omega^{(\ell)}_a \Omega^{(\ell)a} = \Omega^{(\eta)}_a \Omega^{(\eta)a} + \Omega^{(\xi)}_a \Omega^{(\xi)a} \ , \tag{10}$$

with $\Omega^{(\eta)}_a = \eta^b \Omega^{(\ell)}_b \eta_a/\eta$ and $\Omega^{(\xi)}_a = \xi^b \Omega^{(\ell)}_b \xi_a$. We can introduce now a canonical (gravitational) angular momentum as

$$J_{\mathrm{K}} = \frac{1}{8\pi}\int_{\mathscr{S}} \Omega^{(\ell)}_a \eta^a dS \ , \tag{11}$$

where the divergence-free character of η^a together with the transformations properties of $\Omega^{(\ell)}_a$ in (8) guarantee the invariance of J under a rescaling of the null normals. This angular momentum on $\mathscr{S}$ coincides with the Komar one, namely $J_{\mathrm{Komar}} = \frac{1}{8\pi}\int_{\mathscr{S}} \nabla_a \eta_b dS^{ab}$ with $dS^{ab} = \frac{1}{2}(k^a\ell^b - \ell^a k^b)dS$, if η^a can be extended as a Killing vector to a spacetime neighborhood of $\mathscr{S}$.

2.2 Electromagnetic field

Let us consider an electromagnetic field on (M, g_{ab}) with strength field (Faraday) tensor F_{ab}. On a local chart we can express F_{ab} in terms of a vector potential A_a as $F_{ab} = \nabla_a A_b - \nabla_b A_a$. The electromagnetic stress-energy tensor is given by

$$T^{\mathrm{EM}}_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right) . \tag{12}$$

Given $\mathscr{S}$, we denote the electric and magnetic field components normal to $\mathscr{S}$ as

$$E_\perp = F_{ab}\ell^a k^b \quad , \quad B_\perp = {}^*F_{ab}\ell^a k^b \ , \tag{13}$$

where ${}^*F_{ab}$ is the dual of F_{ab}, namely ${}^*F_{ab} = \frac{1}{2}\varepsilon_{abcd}F^{cd}$ with ε_{abcd} the volume element of g_{ab}. Electric and magnetic charges can be expressed as (e.g. [10, 17])

$$Q_{\mathrm{E}} = \frac{1}{4\pi}\int_{\mathscr{S}} E_\perp dS \ , \ \ Q_{\mathrm{M}} = \frac{1}{4\pi}\int_{\mathscr{S}} B_\perp dS \ . \tag{14}$$

When discussing the angular momentum in the presence of an electro-magnetic field, we add $\mathscr{L}_\eta A_a = 0$ to the axisymmetry requirements (9). Then, the following canonical notion of total angular momentum can be introduced on $\mathscr{S}$ [19, 48, 8, 29]

$$J = J_{\mathrm{K}} + J_{\mathrm{EM}} = \frac{1}{8\pi}\int_{\mathscr{S}} \Omega^{(\ell)}_a \eta^a dS + \frac{1}{4\pi}\int_S (A_a\eta^a) E_\perp dS \ . \tag{15}$$

2.2.1 Yang-Mills fields

Given a Yang-Mills theory with Lie group G, the dynamical fields are given in terms of a 1-form A_a evaluated on the Lie algebra $\mathscr{G}$ of G. More properly, A_a is a connection on a principal $G-$bundle P over the spacetime M. Denoting the generators in $\mathscr{G}$ as T_i and writing the Lie-algebra commutation relations as

$$[T_i, T_j] = C^k_{ij} T_k \ , \tag{16}$$

the Cartan-Killing quadratic form on $\mathscr{G}$ is given by

$$\mathrm{k}_{ij} = C^k_{il} C^l_{jk} \ , \tag{17}$$

which is non-degenerate for semisimple Lie algebras. For real compact Lie algebras, k_{ij} is non-degenerate and positive-definite (usually a basis $\{T_i\}$ of $\mathscr{G}$ such that $\mathrm{k}_{ij} = \delta_{ij}$ is employed). More generally, the non-degenerate positive-definite character of k_{ij} holds for Lie groups corresponding to products of compact real Lie groups and $U(1)$ factors. Writing the Yang-Mills connection as $\mathrm{A}_a = A_a{}^i T_i$, the Yang-Mills tensor $\mathrm{F}_{ab} = F_{ab}{}^i T_i$ is given by the curvature of A_a, that is $\mathrm{F}_{ab} = (d\mathrm{A})_{ab} + \mathrm{A}_a \wedge \mathrm{A}_b = \left(\nabla_a A_b{}^k - \nabla_b A_a{}^k + C^k_{ij} A^i_a A^j_b\right) T_k$. The Yang-Mills stress-energy tensor can be written

$$T^{\mathrm{YM}}_{ab} = \frac{1}{4\pi} \mathrm{k}_{ij} \left(F_{ac}{}^i F_b{}^{cj} - \frac{1}{4} g_{ab} F_{cd}{}^i F^{cdj} \right). \tag{18}$$

We can define Yang-Mills electric and magnetic charges [20, 50, 11] as

$$Q^{\mathrm{YM}}_{\mathrm{E}} = \frac{1}{4\pi} \int |\mathrm{E}^{\mathrm{YM}}_{\perp}| \, dS \quad , \quad Q^{\mathrm{YM}}_{\mathrm{M}} = \frac{1}{4\pi} \int |\mathrm{B}^{\mathrm{YM}}_{\perp}| \, dS \ , \tag{19}$$

where

$$|\mathrm{E}^{\mathrm{YM}}_{\perp}| = \left[\left(F_{ab}{}^i k^a \ell^b \right) \mathrm{k}_{ij} \left(F_{cd}{}^j k^c \ell^d \right) \right]^{\frac{1}{2}} \ , \ |\mathrm{B}^{\mathrm{YM}}_{\perp}| = \left[\left({}^*F_{ab}{}^i k^a \ell^b \right) \mathrm{k}_{ij} \left({}^*F_{cd}{}^j k^c \ell^d \right) \right]^{\frac{1}{2}} \tag{20}$$

Electromagnetic theory corresponds to the commutative case $G = U(1)$. In particular, the Yang-Mills principal fiber-bundle perspective sheds light on the topological nature of the magnetic charge Q_{M}, offering an understanding of magnetic monopoles as associated with the non-triviality of the $U(1)$-bundle over M (see e.g. [52, 47]), where Q_{M} is controlled by the first Chern class of the $U(1)$-bundle.

3 Stability of marginally outer trapped surfaces

The stability for marginally trapped surfaces is the crucial element in the present discussion of the area inequalities. This notion is extensively reviewed in the contribution by M. Mars in this volume. We discuss the basic elements here needed.

3.1 Basic definitions

First, we choose conventionally ℓ^a as the *outgoing* null vector at $\mathscr{S}$, and refer to $\mathscr{S}$ as a *marginally outer trapped surface* (MOTS) if $\theta^{(\ell)} = 0$. Note that no condition is required on the ingoing expansion $\theta^{(k)}$. The stability of MOTS is introduced in terms of the deformation operator δ_v on $\mathscr{S}$, that controls the infinitesimal variations of geometric objects defined on $\mathscr{S}$ under an infinitesimal deformation of the surface along a vector v^a on $\mathscr{S}$ (here, v^a will be always normal to $\mathscr{S}$). This operator δ_v, discussed in detail in [3, 4] (see also M. Mars contribution and [16, 18]), is the analogue in the Lorentzian setting to the deformation operator employed in the discussion of minimal surfaces in Riemannian geometry. We require $\mathscr{S}$ to be *stably outermost* in the sense introduced in [3, 4] (see also [37, 46]):

Definition 1. *Given a closed orientable marginally outer trapped surface $\mathscr{S}$ and a vector v^a orthogonal to it, we will refer to $\mathscr{S}$ as* stably outermost with respect to the direction v^a *iff there exists a function $\psi > 0$ on $\mathscr{S}$ such that the variation of $\theta^{(\ell)}$ with respect to ψv^a fulfills the condition $\delta_{\psi v}\theta^{(\ell)} \geq 0$.*

More specifically, we require $\mathscr{S}$ to be *spacetime stably outermost* [43, 30].

Definition 2. *A closed orientable marginally outer trapped surface $\mathscr{S}$ is referred to as* spacetime stably outermost *if there exists an outgoing ($-k^a$-oriented) vector $X^a = \gamma\ell^a - \psi k^a$, with $\gamma \geq 0$ and $\psi > 0$, with respect to which $\mathscr{S}$ is stably outermost:*

$$\delta_X \theta^{(\ell)} \geq 0. \tag{21}$$

If, in addition, X^a (i.e. γ, ψ) and $\Omega_a^{(\ell)}$ are axisymmetric, we will refer to $\delta_X \theta^{(\ell)} \geq 0$ as an (axisymmetry-compatible) spacetime stably outermost condition.

Alternatively, one could introduce the notion of stability for MOTS in terms of the non-negativity of the principal eigenvalue λ_v of the stability operator L_v associated with δ_v, namely $L_v\psi = \delta_{\psi v}\theta^{(\ell)}$. Although L_v is not self-adjoint, its principal eigenvalue (i.e. the eigenvalue with smallest real part) is indeed real. Then, the characterization in Definition 1 can be proved as a lemma [3, 4]. This is the strategy followed in the contribution by M. Mars in this volume.

Finally, note that the characterization of MOTSs as spacetime stably outermost is independent of the choice of future-oriented null normals ℓ^a and k^a. Indeed, given $f > 0$, for $\ell'^a = f\ell^a$ and $k'^a = f^{-1}k^a$ we can write $X^a = \gamma\ell^a - \psi k^a = \gamma'\ell'^a - \psi' k'^a$ (with $\gamma' = f^{-1}\gamma \geq 0$ and $\psi' = f\psi > 0$), and it holds $\delta_X \theta^{(\ell')} = f \cdot \delta_X \theta^{(\ell)} > 0$.

3.2 Integral-inequality characterizations of MOTS stability

The first step in the proofs of area inequalities (1) consists in casting condition (21) as an integral geometric inequality over $\mathscr{S}$. Condition (21) plays, for MOTS in a Lorentzian context, a role analogous to that of the stability condition for minimal

surfaces in Riemannian geometry. In the Riemannian case this refers to the minimization of the area of $\mathscr{S}$ with respect to arbitrary deformations along αs^a, where s^a is the normal to $\mathscr{S}$ in a given 3-slice and α is an arbitrary function on $\mathscr{S}$. In contrast, the stability condition in Definition 2 only states the existence of a positive function ψ (and, secondarily, of $\gamma \geq 0$). The proof of area inequalities involving the angular momentum requires writing (21) as an integral inequality in terms of arbitrary (axisymmetric) functions α. The following lemma [43] provides this[1]

Lemma 1. *Given a closed orientable marginally outer trapped surface $\mathscr{S}$ satisfying the spacetime stably outermost condition for an axisymmetric X^a, then for all axisymmetric functions α on $\mathscr{S}$*

$$\int_{\mathscr{S}} \left[D_a \alpha D^a \alpha + \frac{1}{2}\alpha^2\, {}^2R \right] dS \geq \tag{22}$$
$$\int_{\mathscr{S}} \left[\alpha^2 \Omega_a^{(\eta)} \Omega^{(\eta)a} + \alpha\beta \sigma_{ab}^{(\ell)} \sigma^{(\ell)ab} + G_{ab}\alpha \ell^a (\alpha k^b + \beta \ell^b) \right] dS \,,$$

where $\beta = \alpha\gamma/\psi$. If in addition we assume that the right hand side in the inequality (22) is not identically zero, then $\mathscr{S}$ has a S^2 topology.

Proof. We basically follow the discussion in section 3.3. of [2] (cf. Th. 2.1 in [36] for a similar reasoning, essentially reducing a non time-symmetric case to a time-symmetric one). First, we evaluate $\delta_X \theta^{(\ell)}/\psi$ for $X^a = \gamma \ell^a - \psi k^a$ in Definition 1, with axisymmetric γ and ψ. For this we use (e.g. Eqs. (2.23) and (2.24) in [16])

$$\begin{aligned}
\delta_{\alpha\ell}\theta^{(\ell)} &= \kappa^{(\alpha\ell)}\theta^{(\ell)} - \alpha \left[\sigma_{ab}^{(\ell)} \sigma^{(\ell)ab} + G_{ab}\ell^a k^b + \frac{1}{2}\left(\theta^{(\ell)}\right)^2 \right] \,, \\
\delta_{\beta k}\theta^{(\ell)} &= \kappa^{(\beta k)}\theta^{(\ell)} + {}^2\Delta\beta - 2\Omega_a^{(\ell)} D^a \beta \\
&\quad + \beta \left[\Omega_a^{(\ell)} \Omega^{(\ell)a} - D^a \Omega_a^{(\ell)} - \frac{1}{2}{}^2R + G_{ab}k^a \ell^b - \theta^{(\ell)}\theta^{(k)} \right] \,,
\end{aligned}$$

where $\kappa^{(v)} = -v^a k^b \nabla_a \ell_b$. Imposing $\theta^{(\ell)} = 0$, we can write for $X^a = \gamma \ell^a - \psi k^a$

$$\begin{aligned}
\frac{1}{\psi}\delta_X \theta^{(\ell)} &= -\frac{\gamma}{\psi}\left[\sigma_{ab}^{(\ell)} \sigma^{(\ell)ab} + G_{ab}\ell^a \ell^b \right] \\
&\quad - {}^2\Delta \ln\psi - D_a \ln\psi D^a \ln\psi + 2\Omega_a^{(\ell)} D^a \ln\psi \\
&\quad - \left[-D^a \Omega_a^{(\ell)} + \Omega_c^{(\ell)} \Omega^{(\ell)c} - \frac{1}{2}{}^2R + G_{ab}k^a \ell^b \right] \,.
\end{aligned} \tag{23}$$

We multiply by α^2 for arbitrary (axisymmetric) α and integrate on $\mathscr{S}$. Using $\int_{\mathscr{S}} \frac{\alpha^2}{\psi} \delta_X \theta^{(\ell)} dS \geq 0$ and integrating by parts, we can write

[1] Alternatively, one could start characterizing MOTS stability in terms of the principal eigenvalue λ_X. Then, the expression of λ_X in a Rayleigh-Ritz type characterization leads essentially to the integral inequality. See M. Mars contribution, where the role of α is played by the function u.

$$
\begin{aligned}
0 \leq \quad & \int_{\mathscr{S}} \alpha\beta \left[-\sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} - G_{ab}\ell^a\ell^b\right] dS \\
&+ \int_{\mathscr{S}} \alpha^2 \left[-\Omega^{(\ell)}_a \Omega^{(\ell)a} + \frac{1}{2}{}^2R - G_{ab}k^a\ell^b\right] dS \\
&+ \int_{\mathscr{S}} \left[2\alpha D_a\alpha D^a \ln\psi - \alpha^2 D_a \ln\psi D^a \ln\psi\right] dS \\
&+ \int_{\mathscr{S}} \left[2\alpha^2 \Omega^{(\ell)}_a D^a \ln\psi - 2\alpha\Omega^{(\ell)}_a D^a\alpha\right] dS \,.
\end{aligned} \tag{24}
$$

From the axisymmetry of α and ψ, ${\Omega^{(\eta)}}^a D_a\alpha = {\Omega^{(\eta)}}^a D_a\psi = 0$, and using (10)

$$
\begin{aligned}
0 \leq \quad & \int_{\mathscr{S}} \alpha\beta \left[-\sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} - G_{ab}\ell^a\ell^b\right] dS \\
&+ \int_{\mathscr{S}} \alpha^2 \left[-\Omega^{(\eta)}_a {\Omega^{(\eta)}}^a + \frac{1}{2}{}^2R - G_{ab}k^a\ell^b\right] dS \\
&+ \int_{\mathscr{S}} \Big[2(D^a\alpha)(\alpha D_a \ln\psi - \alpha\Omega^{(\xi)}_a) \\
&-(\alpha D_a \ln\psi - \alpha\Omega^{(\xi)}_a)(\alpha D^a \ln\psi - \alpha{\Omega^{(\xi)}}^a)\Big] dS \,.
\end{aligned} \tag{25}
$$

Making use of the following Young's inequality in the last integral

$$
D^a\alpha D_a\alpha \geq 2D^a\alpha(\alpha D_a \ln\psi - \alpha\Omega^{(\xi)}_a) - |\alpha D \ln\psi - \alpha\Omega^{(\xi)}|^2 \tag{26}
$$

inequality (22) follows for all axisymmetric α. Finally, if the right hand side of (22) does not vanish, the sphericity of $\mathscr{S}$ follows by considering a constant α in (22): it implies a positive value for the Euler characteristic of $\mathscr{S}$.

The proof of the area-charge inequality, resulting from dropping the angular momentum J in (1), requires neither a symmetry assumption nor casting (21) in terms of an arbitrary α. We use the following lemma (slightly generalizing that in [30]).

Lemma 2. *Given a closed orientable marginally outer trapped surface $\mathscr{S}$ satisfying the spacetime stably outermost condition, then the following inequality holds*

$$
\int_{\mathscr{S}} \left[G_{ab}\ell^a k^b + N\left(\sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} + G_{ab}\ell^a\ell^b\right)\right] dS \leq 4\pi(1-g), \tag{27}
$$

where g is the genus of $\mathscr{S}$ and $N = \frac{\gamma}{\psi} \geq 0$. If in addition we assume that the left hand side in the inequality (27) is non-negative and not identically zero, then it follows that $g = 0$ and hence $\mathscr{S}$ has the S^2 topology.

Proof. The proof is slightly more simple than that in Lemma 1. We integrate directly expression (23) over $\mathscr{S}$. On the left hand side we use the stability condition (21). Divergence terms in the right hand side integrate to zero and we rearrange terms as

$$
-(D_a \ln\psi - \Omega^{(\ell)}_a)(D^a \ln\psi - {\Omega^{(\ell)}}^a) = -D_a \ln\psi D^a \ln\psi + 2\Omega^{(\ell)}_a D^a \ln\psi - \Omega^{(\ell)}_c {\Omega^{(\ell)}}^c, \tag{28}
$$

so that the integral is non-positive. From Gauss-Bonnet theorem, we write

$$\int_S \frac{1}{2} {}^2R dS = 4\pi(1-g) \ . \tag{29}$$

Collecting these observations, the inequality (22) follows. If the left hand side of the inequality (22) is non-negative it follows that g can be 0 or 1. If it is not identically zero then $g=0$ and hence $\mathscr{S}$ has the S^2 topology.

3.3 Variants to the stably outermost condition

3.3.1 On an *averaged* outermost stably conditions for MOTS

Inequalities (22) and (27) do not require a *point-like* stability condition. We could consider an (in principle weaker) *averaged* stability condition for MOTS.

Definition 3. *Given a closed orientable MOTS $\mathscr{S}$ we will refer to it as (dipole)* weight-averaged stably outermost *if there exists an outgoing ($-k^a$-oriented) vector $x^a = \gamma\ell^a - k^a$, with $\gamma \geq 0$ such that, for all functions α on $\mathscr{S}$, the variations of $\theta^{(\ell)}$ with respect to $X^a = \alpha x^a$ fulfill the integral condition*

$$\int_{\mathscr{S}} (X^a \ell_a) \delta_X \theta^{(\ell)} dS \geq 0 \ . \tag{30}$$

Proofs of inequalities involving the angular momentum (cf. sections 4 and 6) could start from this averaged condition. Note that $(X^a\ell_a) = \alpha = \text{const}$ provides an *averaged stably outermost* condition, the element needed in proving area-charge inequalities (cf. section 5). Finally, a (2n-moment) *weight-averaged stably outermost* condition could be introduced as $\int_{\mathscr{S}} (X^a\ell_a)^n \ \delta_X \theta^{(\ell)} dS \geq 0$, for integers n.

3.3.2 Towards axisymmetry relaxation

Let η^a be a divergence-free vector on $\mathscr{S}$, with squared-norm $\eta = \eta^a\eta_a$ constant along itself, i.e. $\eta^a D_a \eta = 0$ (fulfilled, in particular, by Killing vectors). As in 2.1.1, we write $q_{ab} = \frac{1}{\eta}\eta_a\eta_b + \xi_a\xi_b$, with $\xi^a\eta_a = \xi^a\ell_a = \xi^a k_a = 0$, $\xi^a\xi_a = 1$, and $\Omega_a^{(\eta)} = \eta^b\Omega_b^{(\ell)}\eta_a/\eta$ and $\Omega_a^{(\xi)} = \xi^b\Omega_b^{(\ell)}\xi_a$, so relations (10) hold. The geometric quantity

$$Q[\eta] = \frac{1}{4\pi}\int_{\mathscr{S}} \frac{1}{\sqrt{\eta}} \Omega_a^{(\ell)} \eta^a dS \ , \tag{31}$$

is well defined on $\mathscr{S}$ in the sense that: i) it does not depend on the normalization of the null normal ℓ^a, and ii) there is no normalization ambiguity related to η^a. The first point follows from the transformation properties (8) of $\Omega_a^{(\ell)}$ under $\ell'^a =$

$f\ell^a$, together with the divergence-free character of $\hat{\eta}^a = \eta^a/\sqrt{\eta}$, i.e. $D_a(\hat{\eta}^a) = \frac{1}{\sqrt{\eta}}D_a\eta^a - \frac{1}{2\eta\sqrt{\eta}}\eta^a D_a\eta = 0$. Regarding the second point, $Q[\eta]$ is defined in terms of $\hat{\eta}^a$, with $\hat{\eta}^a\hat{\eta}_a = 1$. This is the analogue of the 2π-orbit normalization for axial Killing vectors in expressions (11) and (15) for the angular momentum (note that η^a needs not to be axial). We can then adapt the MOTS stability condition:

Definition 4. *Given a closed orientable marginally outer trapped surface $\mathscr{S}$ and a divergence free vector η^a on it, $\mathscr{S}$ is said to be η^a-compatible spacetime stably outermost if there exists an outgoing ($-k^a$-oriented) vector $X^a = -\psi k^a$, with $\psi > 0$ and $\eta^a D_a\psi = 0$, such that the variation of $\theta^{(\ell)}$ with respect to X^a fulfills the condition $\delta_X\theta^{(\ell)} \geq 0$.*

The following lemma holds.

Lemma 3. *Given a closed orientable MOTS $\mathscr{S}$ satisfying the η^a-compatible spacetime stably outermost condition for X^a, then for all α such that $\eta^a D_a\alpha = 0$, it holds*

$$\int_{\mathscr{S}}\left[D_a\alpha D^a\alpha + \frac{1}{2}\alpha^2\, {}^2\!R\right]dS \geq \int_{\mathscr{S}}\left[\alpha^2\left(\Omega_a^{(\eta)}\Omega^{(\eta)a} + G_{ab}\ell^a k^b\right)\right]dS\ , \qquad (32)$$

Proof. The proof proceeds exactly as in Lemma 1. It is straightforward to generalize it for $X^a = \gamma\ell^a - \psi k^a$, $\gamma \geq 0$, so that the shear and the $G_{ab}\ell^a\ell^b$ terms are incorporated.

4 The area-angular momentum inequality

We first state the main result in this section (see [43] for further details):

Theorem 1 (cf. Ref. [43]). *Given an axisymmetric closed orientable marginally outer trapped surface $\mathscr{S}$ satisfying the (axisymmetry-compatible) spacetime stably outermost condition, in a spacetime with non-negative cosmological constant and fulfilling the dominant energy condition, it holds the inequality*

$$A \geq 8\pi|J|\ , \qquad (33)$$

where A and J are the area and gravitational (Komar) angular momentum of $\mathscr{S}$. If equality holds, then $\mathscr{S}$ has the geometry of an extreme Kerr throat sphere and, in addition, if the vector X^a in the stability condition can be found to be spacelike then $\mathscr{S}$ is a section of a non-expanding horizon.

The proof of the area-angular momentum inequality (33) has two parts. The first one is purely geometric and provides the lower bound on the area A

$$A \geq 4\pi e^{\frac{\mathscr{M}-8}{8}}\ , \qquad (34)$$

where $\mathscr{M}$ is a functional on the sphere geometry. The second part solves a variational problem, subject to the constraint of keeping constant the a priori given angular momentum J. In particular, it is shown [1, 33, 31] the existence of a minimum

$$\mathscr{M} \geq \mathscr{M}_0 \ , \quad \text{under the constraint } J \text{ fixed} \tag{35}$$

such that the evaluation of (34) on $\mathscr{M}_0$ leads to inequality (33). Moreover the minimizer is unique, this leading to a rigidity result. We focus here on the first geometric part and refer the reader to the proper references on the variational part [1, 33, 31].

Proof. First, we consider an axisymmetric stably outermost MOTS and apply the result in Lemma 1, namely, we consider inequality (22) where we disregard the positive-definite gravitational radiation shear squared term. Imposing Einstein equation, we also disregard the cosmological constant and matter terms under the assumption of non-negative cosmological constant $\Lambda \geq 0$ and the dominant energy condition (note that $\alpha k^b + \beta \ell^b$ is a non-spacelike vector). Therefore

$$\int_{\mathscr{S}} \left[D_a \alpha D^a \alpha + \frac{1}{2}\alpha^2 \, {}^2\!R \right] dS \geq \int_{\mathscr{S}} \alpha^2 \Omega^{(\eta)}_a \Omega^{(\eta)a} dS. \tag{36}$$

Second, we express this inequality in terms of certain potentials for the geometry of $\mathscr{S}$. Assuming a non-vanishing right hand side in (36) (otherwise (34) is trivial), $\mathscr{S}$ has a spherical topology. On an axisymmetric sphere we can always write [9]

$$ds^2 = q_{ab} dx^a dx^b = e^{\sigma} \left(e^{2q} d\theta^2 + \sin^2\theta d\varphi^2 \right) \ , \tag{37}$$

with σ and q functions on θ satisfying $\sigma + q = c$, where c is a constant. Then $dS = e^c dS_0$, with $dS_0 = \sin\theta d\theta d\varphi$. In addition, the squared norm η of the axial Killing vector $\eta^a = (\partial_\varphi)^a$ is given by

$$\eta = e^{\sigma} \sin^2\theta \ . \tag{38}$$

Choosing $\alpha = e^{c-\sigma/2}$ [31], the evaluation of the left hand side in (36) results in

$$\int_{\mathscr{S}} \left[D_a \alpha D^a \alpha + \frac{1}{2}\alpha^2 \, {}^2\!R \right] dS = e^c \left[4\pi(c+1) - \int_{\mathscr{S}} \left(\sigma + \frac{1}{4} \left(\frac{d\sigma}{d\theta} \right)^2 \right) dS_0 \right] \tag{39}$$

To evaluate the right hand side in (36) we note that, due to the S^2 topology of $\mathscr{S}$, we can always express $\Omega^{(\ell)}_a$ in terms of a divergence-free and an exact form

$$\Omega^{(\ell)}_a = \epsilon_{ab} D^b \tilde{\omega} + D_a \lambda \ , \tag{40}$$

with $\tilde{\omega}$ and λ fixed up to a constant. From the axisymmetry of q_{ab} and $\Omega^{(\ell)}_a$ (functions $\tilde{\omega}$ and λ are then axially symmetric) it follows $\Omega^{(\eta)}_a = \epsilon_{ab} D^b \tilde{\omega}$ and $\Omega^{(\xi)}_a = D_a \lambda$. Before proceeding further, we evaluate the angular momentum J. Writing $\eta^a \Omega^{(\ell)}_a = \eta^a \Omega^{(\eta)}_a = \epsilon_{ab} \eta^a D^b \tilde{\omega}$ and expressing ξ^a as $\xi_b = \eta^{-1/2} \epsilon_{ab} \eta^a$, we have

$$\Omega^{(\ell)}_a \eta^a = \eta^{1/2} \xi^a D_a \tilde{\omega} \ . \tag{41}$$

Plugging this into Eq. (11) (or (15), since $J_{\mathrm{EM}} = 0$) and using (37) we find

$$J = \frac{1}{8}\int_0^\pi 2\eta \frac{d\tilde{\omega}}{d\theta} = \frac{1}{8}\int_0^\pi \frac{d\bar{\omega}}{d\theta} = \frac{1}{8}\left[\bar{\omega}(\pi) - \bar{\omega}(0)\right] , \tag{42}$$

where we have introduced the potential $\bar{\omega}$ as $d\bar{\omega}/d\theta \equiv (2\eta)d\tilde{\omega}/d\theta$. The use of $\bar{\omega}$, rather than $\tilde{\omega}$, permits to control directly the angular momentum in terms of the values of $\bar{\omega}$ at the axis. This is crucial to implement the constraint $J = \text{const}$ in the variational problem. We use $\bar{\omega}$ in the following, rather than $\tilde{\omega}$. Further geometric intuition is gained by noting that, if the axial vector η^a on $\mathscr{S}$ extends to a spacetime neighborhood of $\mathscr{S}$ (something not needed in the present discussion), we can define the *twist* vector of η^a as $\omega_a = \varepsilon_{abcd}\eta^b\nabla^c\eta^d$ and the relation $\xi^a\omega_a = \xi^a D_a\bar{\omega}$ holds. In the vacuum case, a spacetime twist potential ω satisfying $\omega_a = \nabla_a\omega$ can be defined, so that $\bar{\omega}$ and ω coincide on $\mathscr{S}$ up to a constant. Note however that $\bar{\omega}$ on $\mathscr{S}$ can be defined always, even in the presence of matter.

From Eqs. (40) and (37) and the adopted choice for α, we have

$$\alpha^2\Omega^{(\eta)}_a\Omega^{(\eta)a} = \frac{\alpha^2}{4\eta^2}D_a\bar{\omega}D^a\bar{\omega} = \frac{1}{4\eta^2}\left(\frac{d\bar{\omega}}{d\theta}\right)^2 . \tag{43}$$

Plugging this into (36) and using (39) we get

$$8(c+1) \geq \mathscr{M}[\sigma,\bar{\omega}] , \tag{44}$$

with

$$\mathscr{M}[\sigma,\bar{\omega}] = \frac{1}{2\pi}\int_{\mathscr{S}}\left[\left(\frac{d\sigma}{d\theta}\right)^2 + 4\sigma + \frac{1}{\eta^2}\left(\frac{d\bar{\omega}}{d\theta}\right)^2\right] dS_0 . \tag{45}$$

Using these expressions and $A = 4\pi e^c$ leads to inequality (34). This completes the first stage in the proof. In a second stage, by solving the variational problem defined by $\mathscr{M}[\sigma,\bar{\omega}]$ with J constant as a constraint, one can prove [1, 33]

$$\mathscr{M} \geq \mathscr{M}_0 = 8\ln(2|J|) + 8 . \tag{46}$$

This, namely $e^{(\mathscr{M}-8)/8} \geq 2|J|$, together with (34) leads to area-angular momentum inequality (33). Actually, the only minimizer for $\mathscr{M}_0$ in (46) is extremal Kerr, this leading to a rigidity result [31, 43]: if equality in (33) holds, first, the intrinsic geometry of $\mathscr{S}$ is that of an extreme Kerr throat sphere [28] and, second, the vanishing of the positive-definite terms in (22) implies, for spacelike X^a in (21), the vanishing of the shear $\sigma^{(\ell)}_{ab}$ so that $\mathscr{S}$ is an *instantaneous* (non-expanding) isolated horizon [15].

5 The area-charge inequality. Generalizations

Remarkably, if we drop the angular momentum from the inequality (1) , the resulting area-charge inequality requires neither the use of a variational principle nor the assumption of any symmetry.

Theorem 2 (cf. Ref. [30]). *Given an orientable closed orientable marginally outer trapped surface $\mathscr{S}$ satisfying the spacetime stably outermost condition, in a spacetime which satisfies Einstein equations, with non-negative cosmological constant Λ and such that the non-electromagnetic matter fields T_{ab} satisfy the dominant energy condition, then it holds*

$$A \geq 4\pi \left(Q_{\mathrm{E}}^2 + Q_{\mathrm{M}}^2\right), \tag{47}$$

where A, Q_{E} and Q_{M} are the area, electric and magnetic charges of $\mathscr{S}$.

Proof. We start from Lemma 2 and use inequality (27) and Einstein equations (2). Since the vector $k^a + \gamma/\psi \ell^a$ is timelike or null, using that the tensor T_{ab} satisfies the dominant energy condition (and in particular the null energy condition), that Λ is non-negative and the term proportional to N is definite-positive, we get from (27)

$$8\pi \int_{\mathscr{S}} T^{\mathrm{EM}}_{ab} \ell^a k^b dS \leq 4\pi(1-g). \tag{48}$$

The term $T^{\mathrm{EM}}_{ab}\ell^a k^b$ can be written as

$$T^{\mathrm{EM}}_{ab}\ell^a k^b = \frac{1}{8\pi}\left[\left(\ell^a k^b F_{ab}\right)^2 + \left(\ell^a k^b {}^*F_{ab}\right)^2\right]. \tag{49}$$

This result is purely algebraic, something crucial for the later generalization to Yang-Mills fields. To derive (49) we use the decomposition (3) for g_{ab} and calculate

$$F_{ab}F^{ab} = -2\left(\ell^a k^b F_{ab}\right)^2 - 4q^{ab}k^c F_{ac}\ell^d F_{bd} + F_{ab}F_{cd}q^{ac}q^{bd}, \tag{50}$$

and

$$\ell^a k^c F_{ab} F_c{}^b = \left(\ell^a k^b F_{ab}\right)^2 + q^{ab}k^c F_{ac}\ell^d F_{bd}. \tag{51}$$

Noting that the pull-back of F_{ab} on the surface $\mathscr{S}$ is proportional to the volume element ε_{ab} of the surface $\mathscr{S}$, we can evaluate $F_{ab}F_{cd}q^{ac}q^{bd}$ and $\left(\varepsilon^{ab}F_{ab}\right)^2$ to obtain

$$F_{ab}F_{cd}q^{ac}q^{bd} = \frac{1}{2}\left(\varepsilon^{ab}F_{ab}\right)^2 = 2\left({}^*F_{ab}\ell^a k^b\right)^2, \tag{52}$$

where the identity ${}^*F_{ab}\ell^a k^b = \frac{1}{2}F_{ab}\varepsilon^{ab}$ follows from the relation $\varepsilon_{ab} = \varepsilon_{abcd}\ell^c k^d$. Inserting these expressions into Eq. (12) we obtain (49). Then, using relation (49) into inequality (48) we get

$$\int_{\mathscr{S}} \left[\left(\ell^a k^b F_{ab} \right)^2 + \left(\ell^a k^{b*} F_{ab} \right)^2 \right] dS \leq 4\pi(1-g). \tag{53}$$

If the left hand side is identically zero then the charges are zero and the inequality (47) is trivial. We assume that it is not zero at some point and hence we have $g = 0$. To bound the left hand side of inequality (53) we use Hölder inequality on $\mathscr{S}$. For integrable functions f and h, Hölder inequality is given by

$$\int_{\mathscr{S}} fh dS \leq \left(\int_{\mathscr{S}} f^2 dS \right)^{1/2} \left(\int_{\mathscr{S}} h^2 dS \right)^{1/2}. \tag{54}$$

If we take $h = 1$, then we obtain

$$\int_{\mathscr{S}} f dS \leq \left(\int_{\mathscr{S}} f^2 dS \right)^{1/2} A^{1/2}. \tag{55}$$

where A is the area of $\mathscr{S}$. Using this inequality in (53) we finally obtain

$$A^{-1} \left[\left(\int_{\mathscr{S}} \ell^a k^b F_{ab} dS \right)^2 + \left(\int_{\mathscr{S}} \ell^a k^{b*} F_{ab} dS \right)^2 \right] \leq 4\pi. \tag{56}$$

Finally, we use the expression of the charges (14) to express the left-hand-side of (56) in terms of Q_{E} and Q_{M}. Hence the inequality (47) follows.

5.1 Yang-Mills charges

The derivation of the area-charge inequality (47) does not involve Maxwell equations, only the algebraic form of the electromagnetic stress-energy tensor (12) is used. Given the similar structure of the Yang-Mills stress-energy tensor (18), the result generalizes to include Yang-Mills charges (19), for compact Lie groups.

Corollary 1. *Under the conditions of Theorem 2, for a Yang-Mills theory with compact simple Lie group G (more generally with G given by a product of compact simple Lie groups and $U(1)$ factors) it holds the inequality*

$$A \geq 4\pi \left[\left(Q_{\mathrm{E}}^{\mathrm{YM}} \right)^2 + \left(Q_{\mathrm{M}}^{\mathrm{YM}} \right)^2 \right] . \tag{57}$$

Proof. Proceeding exactly as in Theorem 2 and writing

$$T_{ab}^{\mathrm{YM}} \ell^a k^b = \frac{1}{8\pi} \mathrm{k}_{ij} \left[\left(\ell^a k^b F_{ab}{}^i \right) \left(\ell^c k^d F_{cd}{}^j \right) + \left(\ell^a k^{b*} F_{ab}{}^i \right) \left(\ell^c k^{c*} F_{cd}{}^j \right) \right]. \tag{58}$$

we derive the analogue of inequality (53)

$$4\pi \geq \int_{\mathscr{S}} \left[\left(\ell^a k^b F_{ab}{}^i\right) \mathrm{k}_{ij} \left(\ell^c k^d F_{cd}{}^j\right) + \left(\ell^a k^{b*} F_{ab}{}^i\right) \mathrm{k}_{ij} \left(\ell^c k^{c*} F_{cd}{}^j\right) \right] dS \,. \tag{59}$$

In this case, we can write the form (55) of Hölder inequality as

$$\int_{\mathscr{S}} \mathrm{k}_{ij} V^i V^j \geq \frac{1}{A} \left(\int_{\mathscr{S}} \left(\mathrm{k}_{ij} V^i V^j\right)^{\frac{1}{2}} dS \right)^2 \,, \tag{60}$$

for compact Lie algebras, for which k_{ij} in (17) is definite-positive [just take $f^2 = \mathrm{k}_{ij} V^i V^j \geq 0$ in (55)]. Using inequality (60) in (59) leads to inequality (57).

5.2 Further generalizations

The area-charge inequality can be extended to incorporate the quantity $Q[\eta]$ in (31).

Corollary 2. *Under the conditions of Lemma 3, the following inequality holds*

$$A \geq 4\pi Q[\eta]^2 \,. \tag{61}$$

Proof. Starting from inequality (32), choose $\alpha = 1$ and drop the electromagnetic or Yang-Mills components (it is straightforward to include them). Then we can write

$$4\pi \geq \int_{\mathscr{S}} \Omega_a^{(\eta)} \Omega^{(\eta)a} dS = \int_{\mathscr{S}} \frac{1}{\eta} \left(\Omega_a^{(\ell)} \eta^a\right)^2 dS = \int_{\mathscr{S}} \left(\frac{1}{\sqrt{\eta}} \Omega_a^{(\ell)} \eta^a \right)^2 dS \,. \tag{62}$$

Using again inequality (55), now with $f = \frac{1}{\sqrt{\eta}} \Omega_a^{(\ell)} \eta^a$, we obtain

$$\int_{\mathscr{S}} \left(\frac{1}{\sqrt{\eta}} \Omega_a^{(\ell)} \eta^a \right)^2 dS \geq \frac{1}{A} \left(\int_{\mathscr{S}} \frac{1}{\sqrt{\eta}} \Omega_a^{(\ell)} \eta^a dS \right)^2 \,, \tag{63}$$

from which inequality (61) follows when using expression (31) for $Q[\eta]$.

Two remarks are in order. First, inequality (61) does not reduce to the area-angular momentum inequality (33), even if η^a is an axial Killing vector. Even in this case, the quantity $Q[\eta]$ is not an angular momentum due to the $1/\sqrt{\eta}$ factor (this is easily seen on dimensional grounds). However, whenever existing, $Q[\eta]$ is a geometric quantity providing a non-trivial lower bound for the area. Second, the area-charge geometric inequalities (47), (57) and (61) can be collected in the more general form

$$A \geq 4\pi \left[Q_{\mathrm{E}}^2 + Q_{\mathrm{M}}^2 + \left(Q_{\mathrm{E}}^{\mathrm{YM}}\right)^2 + \left(Q_{\mathrm{M}}^{\mathrm{YM}}\right)^2 + Q[\eta]^2 \right] \,, \tag{64}$$

assuming that the individual terms make sense.

5.2.1 The Cosmological constant and stability operator eigenvalue

The area-charge inequality has been extended in Ref. [49] to include the cosmological constant Λ and the principal eigenvalue λ of the stability operator associated with the deformation operator δ (λ is a real number [3, 4]). The inequality reads

$$\Lambda^* A^2 - 4\pi(1-g)A + (4\pi)^2 \sum_i Q_i^2 \leq 0 \;, \tag{65}$$

where $\Lambda^* \equiv \Lambda + \lambda$ and Q_i correspond to Q_{E}, Q_{M}, $Q_{\mathrm{E}}^{\mathrm{YM}}$, $Q_{\mathrm{M}}^{\mathrm{YM}}$ and $Q[\eta]$. The previous inequality (64) follows from the stability condition $\Lambda^* > 0$ and $g = 0$. We highlight the remarkable fact that the cosmological constant and the principal eigenvalue enter formally in exactly the same manner. This suggests the possibility of linking global and quasi-local notions of stability in black hole spacetime geometries.

5.2.2 Energy flux terms

From a physical perspective, it is suggestive to rewrite the previous inequality (65) without dropping neither the matter terms nor the piece proportional to N in (27). Following [13, 14] we define $\mathscr{F}_{\mathrm{grav}} \equiv \frac{1}{16\pi}\int_{\mathscr{S}} N\sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} dS$ as the instantaneous flux of (transverse [38, 39]) gravitational radiation measured by an (Eulerian) observer associated with a foliation with lapse function N. Expressing the flux of matter energy as $\mathscr{F}_{\mathrm{matter}} \equiv \int_{\mathscr{S}} T^{\mathrm{M}}_{ab}\ell^a t^b dS$ (with $t^a = k^b + N\ell^b$ a timelike vector) and the electromagnetic Poynting flux as $\mathscr{F}_{EM} = \int_{\mathscr{S}} NT^{\mathrm{EM}}_{ab}\ell^a\ell^b dS$, we write (with $g = 0$)

$$\frac{1}{2} \geq \frac{\Lambda^*}{2}\left(\frac{A}{4\pi}\right) + \frac{1}{2}\left(\frac{4\pi}{A}\right)\sum_i Q_i^2 + \mathscr{F}_{\mathrm{EM}} + \mathscr{F}_{\mathrm{matter}} + 2\mathscr{F}_{\mathrm{grav}} \tag{66}$$

This emphasizes the role of integral inequalities (22), (27) and (32) in Lemmas 1, 2 and 3 as *energy flux* inequalities. In particular, flux inequality (66) indicates that the instantaneous flux of energy into a stable black hole horizon is bounded from above so that it cannot be arbitrarily large.

6 The area-angular momentum-charge inequality

After discussing the area-angular momentum and area-charge inequalities, we address now the inequality incorporating all relevant quantities in Einstein-Maxwell theory.

Theorem 3 (cf. Refs. [34, 35]). *Given an axisymmetric closed orientable marginally outer trapped surface $\mathscr{S}$ satisfying the (axisymmetry-compatible) spacetime stably outermost condition, in a spacetime with non-negative cosmological constant and matter content fulfilling the dominant energy condition, it holds the inequality*

$$(A/(4\pi))^2 \geq (2J)^2 + (Q_E^2 + Q_M^2)^2 \tag{67}$$

where A is the area of $\mathscr{S}$ and J, Q_E and Q_M are, respectively, the total (gravitational and electromagnetic) angular momentum, the electric and the magnetic charges associated with $\mathscr{S}$. If equality holds, then $\mathscr{S}$ has the geometry of an extreme Kerr-Newman throat sphere and, in addition, if vector X^a in the stability condition can be found to be spacelike then $\mathscr{S}$ is a section of a non-expanding horizon.

Proof. The proof [34, 35] follows the steps in Theorem 1, namely with a first stage in which a lower bound (34) on the area is derived, followed by the resolution of a variational problem under the constraints of keeping J, Q_E and Q_M fixed.

First, starting from inequality (22), proceeding then as in the derivation of (36) and using relations (49) and (13), we obtain

$$\int_{\mathscr{S}} \left[|D\alpha|^2 + \frac{1}{2}\alpha^2\, {}^2R\right] dS \geq \int_{\mathscr{S}} \alpha^2 \left[|\Omega^{(\eta)}|^2 + (E_\perp^2 + B_\perp^2)\right] dS \, . \tag{68}$$

From this expression, contact can be made [33] with the proof in [41] to establish inequality (67) for vanishing Q_M. Here, we rather follow [34, 35] the strategy in section 4. In order to identify the relevant action functional $\mathscr{M}$ for the variational problem, in particular its dependence on appropriate potentials permitting to control the constraints on J, Q_E and Q_M, we adopt again a coordinate system (37) on the axisymmetric sphere and use the decomposition (40) introducing the potential $\tilde{\omega}$. From expressions (14) for Q_E and Q_M and (15) for J, we write (see details in [35])

$$\begin{aligned}
Q_E &= \frac{1}{2}\int_0^\pi E_\perp e^c \sin\theta d\theta = \frac{1}{2}[\psi(\pi) - \psi(0)] \\
Q_M &= \frac{1}{4\pi}\int_0^\pi \frac{dA_\varphi}{d\theta} d\theta = \frac{1}{2}[\chi(\pi) - \chi(0)] \\
J &= \frac{1}{8}\int_0^\pi \left(2\eta\frac{d\tilde{\omega}}{d\theta} + 2\chi\frac{d\psi}{d\theta} - 2\psi\frac{d\chi}{d\theta}\right) = \frac{1}{8}[\omega(\pi) - \omega(0)] \, ,
\end{aligned} \tag{69}$$

where we have introduced the new potentials ω, χ and ψ on $\mathscr{S}$

$$\begin{aligned}
\frac{d\psi}{d\theta} &= E_\perp e^c \sin\theta \quad , \quad \chi = A_\varphi \, , \\
\frac{d\omega}{d\theta} &= 2\eta\frac{d\tilde{\omega}}{d\theta} + 2\chi\frac{d\psi}{d\theta} - 2\psi\frac{d\chi}{d\theta} = \frac{d\bar{\omega}}{d\theta} + 2\chi\frac{d\psi}{d\theta} - 2\psi\frac{d\chi}{d\theta} \, .
\end{aligned} \tag{70}$$

Therefore fixing ω, χ and ψ on the axis does control the values of Q_E and Q_M and J in the variational problem. Using these potentials in (68), with $\alpha = e^{c-\sigma/2}$, we get

$$8(c+1) \geq \mathscr{M}[\sigma, \omega, E_\perp, A_\varphi] \, , \tag{71}$$

where

$$\mathscr{M}[\sigma, \omega, \psi, \chi] = \frac{1}{2\pi}\int_{\mathscr{S}} \left[4\sigma + |D\sigma|^2\right. \tag{72}$$

$$+\frac{|D\omega - 2\chi D\psi + 2\psi D\chi|^2}{\eta^2} + \frac{4}{\eta}(|D\psi|^2 + |D\chi|^2)\Big] dS_0 \ ,$$

from which an inequality (34) is recovered by using $A = 4\pi e^c = 4\pi e^{\sigma(0)}$. The proof of the area-charge-angular momentum inequality (67) is completed by showing that

$$\mathscr{M} \geq \mathscr{M}_0 = 8 \ln \sqrt{(2J)^2 + (Q_{\rm E}^2 + Q_{\rm M}^2)^2} + 8 \ , \tag{73}$$

under the constraint of keeping J, $Q_{\rm E}$ and $Q_{\rm M}$ fixed. Here $\mathscr{M}_0$ corresponds to the evaluation of $\mathscr{M}$ on extremal Kerr-Newman with J, with $Q_{\rm E}$ and $Q_{\rm M}$ given. The details of this variational problem are discussed in [35], where rigidity is also proved.

7 Discussion

We have reviewed a set of geometric inequalities holding for stably outermost marginally trapped surfaces embedded in generic dynamical, non-necessarily axisymmetric spacetimes with ordinary matter that can extend and cross the black hole horizon. These inequalities provide lower bounds for the area A, in terms of expressions involving (linearly) the angular momentum J and (quadratically) the electric and magnetic charges, $Q_{\rm E}$ and $Q_{\rm M}$. Extensions including Yang-Mills charges, $Q_{\rm E}^{\rm YM}$ and $Q_{\rm M}^{\rm YM}$, as well as a charge $Q[\eta]$ for certain divergence-free vectors, have also been discussed. If J is present, axisymmetry is required on the surface (and only on the surface). Otherwise the inequalities involve no symmetry requirements.

We have adopted a purely quasi-local spacetime Lorentzian approach. However, it is worthwhile to note that these inequalities were initially discussed on initial data in spatial 3-slices by using Riemannian techniques, in particular minimal surfaces. Although more stringent in their spacetime requirements, whenever applicable, such versions also hold on more general surfaces that marginally outer trapped surfaces. We have however focused here on the specific context of black hole horizons. In this setting, the adoption of a spacetime perspective based entirely on purely Lorentzian concepts has offered crucial geometric insights into the problem: all geometric elements in the proof acquire a clear spacetime meaning. This has lead to a refinement in the required conditions permitting, in particular, the generic incorporation of matter in the discussion. The crucial ingredient enabling the shift to a purely Lorentzian discussion has been the identification of the stably outermost condition for marginally outer trapped surfaces as the elementary involved notion. In essence, this is the only required ingredient. In this sense, the fulfillment of inequalities (1) is just a fundamental and direct (irreducible) consequence of the Lorentzian structure of spacetime. This is the main conclusion that we want to stress in these notes.

Strictly speaking, the inclusion of the angular momentum in the inequalities requires two further (related) elements: axisymmetry on the surface and an analytical variational principle. This is in contrast with inequalities in which J is absent, that are straightforward geometric consequences of the stability condition. Certainly, the

identification of potentials σ, ω, χ and ψ for the functional $\mathscr{M}$ is related to the spherical topology of $\mathscr{S}$, ultimately controlled by the stability condition for MOTS. However, it is indeed of relevance to assess the role of the axisymmetry and variational treatment requirements in this problem. First, relaxing the local axisymmetry in the angular momentum characterization is of interest in astrophysical contexts. Second, the success of the variational problem is intrinsically related to the existence of particular spacetimes, namely extremal stationary (axisymmetric) black holes, that saturate the inequality and simultaneously provide a (unique) minimum for the functionals $\mathscr{M}$. A better understanding of the structural role played by the variational principle in the proofs could offer insight into the properties of the space of solutions of the theory. In particular, the observation that spacetimes admitting symmetries are singular points in the space of solution of (vacuum) Einstein equations [7, 32] (namely conical singularities) could shed some light on the relation between the presence of symmetries and the need of a variational principle.

From a physical perspective, stable marginally trapped surfaces are sections of quasi-local models for black hole horizons. More precisely, the spacetime stably outermost condition is essentially the *outer condition* introduced in [37] for trapping horizons, namely worldtubes of apparent horizons. From an initial data perspective, the (strictly) stably outermost condition is precisely the condition that guarantees the evolution of an initial apparent horizon into a dynamical horizon [3, 4] with a unique foliation by marginally outer trapped surfaces [12]. The inequalities here studied provide a characterization of the notion of black hole horizon (sub)extremality [17]. Moreover, the rigidity results imply that the saturation of the inequalities characterize the extremality of the horizon geometry. These considerations endorse the discussion of the first law of thermodynamics in dynamical horizons [13, 14] where, in particular, the positivity of the surface gravity is equivalent to the fulfillment of the inequalities here discussed. Equivalently, support is given for the physical validity of the Christodoulou mass, as a function growing with the area (for fixed J and Q_i). Beyond the inequalities among A, J and the charges Q_i, but still in the context of energy balance equations, we have noted in section 5.2.2 that the integral characterization of the stability condition can be interpreted as an *energy flux* inequality.

In the general context of the standard picture of gravitational collapse [45], the inequalities here studied provide a set of quasi-local geometric probes into black hole dynamics in generic situations. In this sense, it is of interest to explore a possible connection between these inequalities and aspects of the cosmic censorship conjecture (e.g. through their link to related global inequalities [35]), or possible implications in the understanding of partial problems in black hole stability.

We would like to conclude by emphasizing that these inequalities represent a particular example of the extension to a Lorentzian setting of tools and concepts employed in the discussion of minimal surfaces in a Riemannian context. In this sense, this family of problems provides a concrete bridge between research in Riemaniann and Lorentzian geometries.

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References

1. Acena, A., Dain, S., Clement, M.E.G.: Horizon area–angular momentum inequality for a class of axially symmetric black holes. Class. Quant. Grav. **28**, 105,014 (2011). DOI 10.1088/0264-9381/28/10/105014
2. Andersson, L., Eichmair, M., Metzger, J.: Jang's equation and its applications to marginally trapped surfaces (2010)
3. Andersson, L., Mars, M., Simon, W.: Local existence of dynamical and trapping horizons. Phys. Rev. Lett. **95**, 111,102 (2005)
4. Andersson, L., Mars, M., Simon, W.: Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes. arXiv:0704.2889 (2008)
5. Ansorg, M., Hennig, J., Cederbaum, C.: Universal properties of distorted Kerr-Newman black holes. Gen. Rel. Grav. **43**, 1205–1210 (2011). DOI 10.1007/s10714-010-1136-8
6. Ansorg, M., Pfister, H.: A universal constraint between charge and rotation rate for degenerate black holes surrounded by matter. Class. Quant. Grav. **25**, 035,009 (2008). DOI 10.1088/0264-9381/25/3/035009
7. Arms, J., Marsden, J., Moncrief, V.: The structure of the space of solutions of Einstein's equations. II. Several Killing fields and the Einstein Yang-Mills equations. Annals Phys. **144**, 81–106 (1982). DOI 10.1016/0003-4916(82)90105-1
8. Ashtekar, A., Beetle, C., Lewandowski, J.: Mechanics of rotating isolated horizons. Phys.Rev. **D64**, 044,016 (2001). DOI 10.1103/PhysRevD.64.044016
9. Ashtekar, A., Engle, J., Pawlowski, T., Van Den Broeck, C.: Multipole moments of isolated horizons. Class. Quant. Grav. **21**, 2549 (2004)
10. Ashtekar, A., Fairhurst, S., Krishnan, B.: Isolated horizons: Hamiltonian evolution and the first law. Phys. Rev. **D62**, 104,025 (2000)
11. Ashtekar, A., Fairhurst, S., Krishnan, B.: Isolated horizons: Hamiltonian evolution and the first law. Phys.Rev. **D62**, 104,025 (2000). DOI 10.1103/PhysRevD.62.104025
12. Ashtekar, A., Galloway, G.J.: Some uniqueness results for dynamical horizons. Adv. Theor. Math. Phys. **9**, 1 (2005)
13. Ashtekar, A., Krishnan, B.: Dynamical horizons: Energy, angular momentum, fluxes and balance laws. Phys. Rev. Lett. **89**, 261,101 (2002)
14. Ashtekar, A., Krishnan, B.: Dynamical horizons and their properties. Phys. Rev. D **68**, 104,030 (2003)
15. Ashtekar, A., Krishnan, B.: Isolated and dynamical horizons and their applications. Liv. Rev. Relat. **7**, 10 (2004). URL `http://www.livingreviews.org/lrr-2004-10`. URL (cited on 28 January 2008): `http://www.livingreviews.org/lrr-2004-10`
16. Booth, I., Fairhurst, S.: Isolated, slowly evolving, and dynamical trapping horizons: geometry and mechanics from surface deformations. Phys. Rev. **D75**, 084,019 (2007)
17. Booth, I., Fairhurst, S.: Extremality conditions for isolated and dynamical horizons. Phys. Rev. **D77**, 084,005 (2008). DOI 10.1103/PhysRevD.77.084005
18. Cao, L.M.: Deformation of Codimension-2 Surface and Horizon Thermodynamics. JHEP **1103**, 112 (2011). DOI 10.1007/JHEP03(2011)112
19. Carter, B.: Republication of: Black hole equilibrium states part ii. general theory of stationary black hole states. General Relativity and Gravitation **42**, 653–744 (2010). URL `http://dx.doi.org/10.1007/s10714-009-0920-9`. 10.1007/s10714-009-0920-9

20. Chrusciel, P., Kondracki, W.: Some global charges in classical Yang-Mills theory. Phys.Rev. **D36**, 1874–1881 (1987). DOI 10.1103/PhysRevD.36.1874
21. Chruściel, P.T.: Mass and angular-momentum inequalities for axi-symmetric initial data sets I. Positivity of mass. Annals Phys. **323**, 2566–2590 (2008). DOI 10.1016/j.aop.2007.12.010
22. Chruściel, P.T., Li, Y., Weinstein, G.: Mass and angular-momentum inequalities for axi-symmetric initial data sets. II. Angular-momentum. Annals Phys. **323**, 2591–2613 (2008). DOI 10.1016/j.aop.2007.12.011
23. Chrusciel, P.T., Lopes Costa, J.: Mass, angular-momentum, and charge inequalities for axisymmetric initial data. Class. Quant. Grav. **26**, 235,013 (2009). DOI 10.1088/0264-9381/26/23/235013
24. Costa, J.L.: A Dain Inequality with charge. arXiv:0912.0838 (2009)
25. Dain, S.: Angular-momentum-mass inequality for axisymmetric black holes. Phys. Rev. Lett. **96**, 101,101 (2006)
26. Dain, S.: Proof of the (local) angular momemtum-mass inequality for axisymmetric black holes. Class. Quant. Grav. **23**, 6845–6856 (2006). DOI 10.1088/0264-9381/23/23/015
27. Dain, S.: Proof of the angular momentum-mass inequality for axisymmetric black holes. J. Diff. Geom. **79**, 33–67 (2008)
28. Dain, S.: Extreme throat initial data set and horizon area-angular momentum inequality for axisymmetric black holes. Phys. Rev. **D82**, 104,010 (2010). DOI 10.1103/PhysRevD.82.104010
29. Dain, S.: Geometric inequalities for axially symmetric black holes. arXiv:1111.3615 (2011)
30. Dain, S., Jaramillo, J.L., Reiris, M.: Area-charge inequality for black holes. arXiv:1109.5602 (2011).
31. Dain, S., Reiris, M.: Area - Angular momentum inequality for axisymmetric black holes. Phys.Rev.Lett. **107**, 051,101 (2011). DOI 10.1103/PhysRevLett.107.051101
32. Fischer, A., Marsden, J., Moncrief, V.: The structure of the space of solutions of Einsteins equations I. One Killing field. Ann. Inst. H. Poincaré **33**, 147–194 (1980)
33. Gabach Clement, M.E.: Comment on *Horizon area- Angular momentum inequality for a class of axially symmetric black holes*. arXiv:1102.3834 (2011)
34. Gabach Clement, M.E., Jaramillo, J.L.: Black hole Area-Angular momentum-Charge inequality in dynamical non-vacuum spacetimes. arXiv:1111.6248 (2011)
35. Gabach Clement, M.E., Jaramillo, J.L., Reiris, M.: *in preparation* (2011)
36. Galloway, G.J., Schoen, R.: A Generalization of Hawking's black hole topology theorem to higher dimensions. Commun.Math.Phys. **266**, 571–576 (2006). DOI 10.1007/s00220-006-0019-z
37. Hayward, S.: General laws of black-hole dynamics. Phys. Rev. D **49**, 6467 (1994)
38. Hayward, S.: Energy conservation for dynamical black holes. Phys. Rev. Lett. **93**, 251,101 (2004)
39. Hayward, S.A.: Energy and entropy conservation for dynamical black holes. Phys. Rev. D **70**, 104,027 (2004)
40. Hennig, J., Ansorg, M., Cederbaum, C.: A universal inequality between the angular momentum and horizon area for axisymmetric and stationary black holes with surrounding matter. Class. Quantum Grav. **25**, 162,002 (2008)
41. Hennig, J., Cederbaum, C., Ansorg, M.: A universal inequality for axisymmetric and stationary black holes with surrounding matter in the Einstein-Maxwell theory. Commun. Math. Phys. **293**, 449–467 (2010). DOI 10.1007/s00220-009-0889-y
42. Hollands, S.: Horizon area-angular momentum inequality in higher dimensional spacetimes. arXiv:1110.5814 (2011)
43. Jaramillo, J.L., Reiris, M., Dain, S.: Black hole Area-Angular momentum inequality in non-vacuum spacetimes. Phys.Rev. **D84**, 121,503 (2011). DOI 10.1103/PhysRevD.84.121503
44. Penrose, R.: Gravitational collapse: The role of general relativity. Riv. Nuovo Cim. **1**, 252 (1969)
45. Penrose, R.: Naked singularities. Annals N. Y. Acad. Sci. **224**, 125 (1973)
46. Racz, I.: A simple proof of the recent generalisations of Hawking's black hole topology theorem. Class. Quant. Grav. **25**, 162,001 (2008). DOI 10.1088/0264-9381/25/16/162001

47. Ryder, L.: Dirac monopoles and the Hopf map S(3) to S(2). J.Phys.A **A13**, 437–447 (1980). DOI 10.1088/0305-4470/13/2/012
48. Simon, W.: Gravitational field strength and generalized Komar integral. Gen.Rel.Grav. **17**, 439 (1985). DOI 10.1007/BF00761903
49. Simon, W.: Comment on 'Area-charge inequality for black holes', arXiv:1109.5602. arXiv:1109.6140 (2011)
50. Sudarsky, D., Wald, R.M.: Extrema of mass, stationarity, and staticity, and solutions to the Einstein Yang-Mills equations. Phys.Rev. **D46**, 1453–1474 (1992). DOI 10.1103/PhysRevD.46.1453
51. Wald, R.M.: General Relativity. Chicago University Press (1984)
52. Wu, T.T., Yang, C.N.: Concept of nonintegrable phase factors and global formulation of gauge fields. Phys. Rev. D **12**, 3845–3857 (1975). DOI 10.1103/PhysRevD.12.3845. URL `http://link.aps.org/doi/10.1103/PhysRevD.12.3845`